Homotopy and statistics: of spin and diagonals



1993 J. Phys. A: Math. Gen. 26 L1093

(http://iopscience.iop.org/0305-4470/26/20/006)

J. Phys. A: Math. Gen. 26 (1993) L1093–L1097. Printed in the UK

LETTER TO THE EDITOR

# Homotopy and statistics: of spin and diagonals

L S Schulman

† Physics Department, Clarkson University, Potsdam, NY 13699-5820, USA

Received 6 August 1993

**Abstract.** The path integral treatment of statistics for identical particles is extended to the case where they possess spin by virtue of a continuous internal coordinate. In this case the restriction to bosons or fermions begins at space dimension two. The role of the 'diagonal' (points of coordinate coincidence) in these arguments is clarified.

Because of the natural way that homotopy considerations enter the path integral, this framework has contributed to the understanding of boundary conditions, topology, and gauge fields and their role in producing well defined unitary evolution operators from Hamiltonians that do not possess unique self-adjoint extensions. In this article we address two issues related to the homotopy cum path integral arguments for quantum statistics. One of these is spin. As remarked in [1], [2] deals only with scalar particles. Not only is this a severe physical restriction, but a more ambitious program is thereby thwarted, namely, the idea that there could come from this an intuitive or topological understanding of the spin-statistics theorem. (We do not achieve that goal here either and we expect relativistic considerations to be needed). The other issue we discuss is the exclusion in [2] of coincidence points (now known as the 'diagonal'), and which has excited speculation over the years. It is comprehensively dealt with in [3]. In the treatment we give here, there is no 'diagonal' to exclude in the basic coordinate space; nevertheless, the treatment of [3] is of interest.

Let a system have Hamiltonian $H$, Lagrangian $L$, and be defined on a coordinate space $M$. If $M$ is homotopically non-trivial, the propagator ($x, y \in M$) can be written [4]

$$G(x, t; y) = \sum_{\alpha} e^{i\phi(\alpha)} \sum_{\xi \in \alpha} e^{iS(\xi)} \tag{1}$$

where $\alpha$ labels homotopy classes, and the paths $\xi$ in the second sum are in $\alpha$. $S$ is the classical action. For $x = y$, a concise statement [5, 2] of the restriction on $\{\phi(\alpha)\}$ is that they provide an Abelian (additive) representation of the fundamental homotopy group $\pi_1(M)$.

In [2], the coordinate space for $N$ identical spinless particles is taken to be $(\mathscr{R}^{dN} - \Delta)/\sim$, where $d$ is the dimension of space, $\Delta$ the 'diagonal', namely $N$-tuplets of $d$-vectors on which two or more $d$-vectors coincide, and the set is taken modulo the equivalence '$\sim$' under which two $N$-tuplets are equivalent if they differ by

0305-4470/93/201093 + 05 \$07.50

exchange(s). For $d \geqslant 3$, $\pi_1$ of this space is the permutation group whose only Abelian representations yield fermions and bosons. For $d=2$, $\pi_1$ is the braid group and one has the possibility of parastatistics. This has been discussed under the rubric 'anyons' [6].

As a result of the subtraction of the diagonal in [2], there arose puzzles about identical particles—their collisions, overlap etc. To some extent these were semantic questions, since the behaviour of the wavefunction was never in doubt. With the coordinate space definition that we are about to give, there is nothing to subtract until one climbs to an appropriate covering space where the 'subtraction' takes care of itself. Once the diagonal shows up on the covering space, the arguments of [3] can be invoked.

Our definition uses the fact that for truly identical particles it is meaningless to say who sits where. All that can be given is a list of positions. We therefore take as our coordinate space for $N$ particles, sets of $N$ $d$-vectors. For a set, the order is irrelevant and by definition the points are distinct—otherwise there would be $N-1$ (or fewer) elements in the set. Call the space of these sets $Q$. Open sets of $Q$ correspond to appropriate open sets in $\mathfrak{R}^{dN}$. A metric is induced in the same way.

To build a propagator on $Q$ we follow the prescription of [4]: Go to $Q^*$, the covering space of $Q$, and project. For $d>3$, $Q^*$ is precisely $\mathfrak{R}^{dN}-\Delta$. From this point one proceeds as in [2] and we do not elaborate. For $d=2$ the covering space is larger, but again the missing diagonal makes its (non) appearance at this stage.

One could also view this from the perspective of the fundamental domain for the coordinate space, for example $N$-tuples with a particular ordering. Here one would encounter the self-adjoint extension problem described in [3] and could use their techniques to reach appropriate conclusions on the behaviour of the wavefunction on the diagonal, a property specified in defining the domain of the (extended) Hamiltonian.

Another way to approach the homotopy/path integral arguments is the use of gauge transformations involving functions that are single-valued on the covering space but multivalued on the fundamental domain [7]. For the discrete transformation associated with permutation symmetry we adopt a step function, giving rise to $\delta$ function valued gauge fields. Consider the two-particle case for which one can adopt as fundamental domain $\{(r_1, r_2) \in \mathfrak{R}^6 \,|\, x_1 > x_2\}$ (with $r=(x, y, z)$). On the covering space the function $\pi\Theta(x_1-x_2)$ is single-valued. The resulting gauge field is $\pi\delta(x_1-x_2)$. This is not the same as the $\delta$-function potential since the gauge field appears in the Lagrangian as $\pi\dot{r}\delta(x_1-x_2)$. In ordinary quantum mechanics this field presents no problem. On a line, $u \in \mathfrak{R}$, the Hamiltonian becomes $(p-\mathrm{i}\pi\delta(u))^2/2m$, and from a solution $\phi(u)$ of the Schrödinger equation without the gauge field one can generate a solution with the gauge field by taking $\exp(\mathrm{i}\pi\Theta(u))\phi(u)$.

The homotopy/path integral/statistics arguments can be extended to particles with spin by finding a representation in which such particles have (one-component) wavefunctions taking values in $\mathscr{C}$. This can be done by enlarging the coordinate space; the same approach that allows an ordinary path integral for spin allows conclusions about statistics to be drawn as well.

For a single particle the appropriate coordinate space is $\mathfrak{R}^3 \times \mathrm{SO}(3)$. Because of the non-trivial covering of SO(3) by SU(2) one can get integral or half integral spin [4, 1]. (This approach can also be taken for relativistic particles [8], but because of complications in the Lagrangian we here discuss only the non-relativistic case.) If the single particle lives in two space dimensions, the 'spin' coordinate can take values in SO(2).

For $N$ particles the coordinate space consists of sets of $N$ elements, each of the form $(r, R)$, with $r \in \mathfrak{R}^3$, $R \in SO(3)$. The considerations of the previous section are complicated by the fact that the covering space has $2^N N!$ copies of the fundamental domain, the extra factor $2^N$ coming from the covering of SO(3) by SU(2). The 'diagonal', consists of those $N$-tuples in which (at least) two points map onto the same $\xi$. (Thus, for $N=2$, $((r_1, U), (r_1, -U))$ is on the diagonal.)

To characterize the fundamental homotopy group we introduce more detailed notation. Let

$$Q_N = \{\{\xi_1, \ldots, \xi_N\} | \text{each } \xi = (r, R), \text{with } r \in \mathfrak{R}^d \& R \in SO(d)\} \qquad d = 2, 3.$$

Again, to say that each set in the collection $Q_N$ has $N$ elements means in effect that its points are distinct objects. As above, $Q_N$ is given a topology and metric from its embedding in $(\mathfrak{R}^d \otimes SO(d))^N$. Let $S_{jk}$ be the exchange of $\xi_j$ and $\xi_k$. By an 'exchange' we mean an equivalence class of closed loops of the form $\{\xi_1(t), \ldots, \xi_N(t)\}$, with $\xi_l(t) = \xi_l$ for $l \neq j$, $k$, $\xi_j(0) = \xi_j$, $\xi_k(0) = \xi_k$, $\xi_j(1) = \xi_k$, $\xi_k(1) = \xi_j$ and arranged in such a way that there are no coincidence points for $0 \leqslant t \leqslant 1$. For simplicity we discuss $\pi_1(Q_N)$ in terms of transformations on $Q_N^*$. The effect of the transformation $S_{12}$ is

$$((r_1, U_1), (r_2, U_2)) \rightarrow ((r_2, U_2), (r_1, U_1))$$

where $U \in SU(2)$ is in the covering space of SO(3). (For $d=2$ the definition is essentially the same, but the covering space of SO(2) is $\mathfrak{R}$. The distinctive feature of the $d=2$, spinless, case is that the names of the points exchanged does not specify the homotopy class. This does not effect our forthcoming discussion.) For the transformations that arise from the SO(3) covering by SU(2) we use the notation $L_k$. Thus $L_k$ is the transformation $(r_k, U_k) \rightarrow (r_k, -U_k)$. It is immediate that

$$L_k S_{kj} = S_{kj} L_j. \tag{2}$$

It is also clear that the $L_k$ commute with one another as well as with $S_{mn}$ with $m \neq k$ and $n \neq k$. (For the $d=2$ case similar relations hold but the corresponding $L_k$ sends $\theta_k \in \mathfrak{R}$ to $\theta_k + 2\pi$.)

For the $d=3$ case, what we have called exchanges give rise to the symmetric group as a subgroup of $\pi_1(Q_N)$. The Abelian representations of this group define, as usual, either fermions or bosons. If we now consider representations of the larger group, for each $L_k$ we could have a choice of representation by the two representations of $Z_2$. For $Z_2 = \{-1, +1\}$ and $L \in Z_2$, these representations are $D_0(L) = 1$ or $D_1(L) = L$. However, from (2) we see that for all $k \leqslant n$ the same representation of $Z_2$ must be used. In this way our analysis obtains the desired results, namely, that one looks to the Abelian representations of $\mathcal{S}_N$, yielding only fermions and bosons. As far as we can tell, there is, at this level, no restriction connecting the phases associated with spin and those for statistics. Presumably this would require a relativistic framework.

For $d=2$, in the absence of an internal continuous coordinate the fundamental homotopy group is *not* the symmetric group on $N$ objects. There are paths that exchange particles twice that do not reduce to null paths, and the fundamental group is the braid group. This can give rise to parastatistics. For the case we now consider there is an additional internal coordinate giving rise to the spin. With this degree of freedom, exchange paths can slide past each other—simply change the value of the internal coordinate when the space coordinates coincide. Spinning two-dimensional objects, where that spin arises from a continuous internal coordinate, thus lose the possibility of parastatistics. They may, however, have a non-trivial phase under $2\pi$ rotation.

Several physical systems have been considered candidates for the possession of anyon excitations. However, a spin degree of freedom for the relevant excitation does not necessarily mean that our result rules out parastatistics. Just as the two-dimensionality of the systems is an idealization, based on the freezing out of the third spatial degree of freedom, so the spin degree of freedom may be frozen out. Moreover, even if it still has the possibility of a spin flip, this does not mean that one can associate with this flip a continuous internal coordinate.

We thus find that particles with intrinsic spin realized through an intrinsic top coordinate can have only bose or fermi statistics. For a single spatial dimension, with a one-dimensional internal coordinate, codimension counting for the exchange again affords the same opportunity for parastatistics as the dimension two, no continuous internal coordinate.

The relation between the internal coordinate (SO(3), SO(2) etc.) and the more common spin formalism is essentially that of projection [4, 9, 10].

We have extended the path integral statistics-via-homotopy arguments to the case of particles with spin. When that spin arises from a continuous internal variable, particle statistics are restricted to the usual boson or fermions. With the extra internal degree of freedom, this restriction applies to space dimension two as well. Only in one space dimension, with an internal one-dimensional degree of freedom, could parastatistics emerge. (For two-dimensional particles lacking an internal continuous degree of freedom, the usual anyon possibilities remain.) It should also be noted that the same arguments apply for any 'internal' quantum number arising from internal continuous motion. The reasoning given for SO($d$) would go through unchanged. We have also given a natural prescription for the identical particle coordinate space for which no 'subtraction of the diagonal' need be invoked. With this definition, it is only when climbing to the covering space (to produce the propagator using the path integral) that one excludes the so-called diagonal. If at this point one chose to drop back to the fundamental domain, then the self-adjoint extension techniques of [3] could be used.

There are three matters, not addressed here, that come to mind. First, when is a degree of freedom 'frozen'? Discussion of particles in two dimensions generally involve an idealization in which the possibility of motion in the third dimension is effectively frozen out. Nevertheless, statements based on this approximate topology, for example the possibility of parastatistics, are valid. Similar questions can arise in discussing the Aharonov–Bohm effect, when one idealizes the impenetrable solenoid as simply absent, and goes on to examine the consequences of multiple connectivity. In that case it is the breakdown of the idealization that fixes the actual phase allowed by the topological idealization. These matters were taken up in [5]. A second question has to do with the possibility of associating a continuous internal variable with a spin degree of freedom. There is a long history of making such models, notably for spin degrees of freedom [9, 10], but this has mostly seemed a matter of personal preference. In the case of statistics for particles in two space dimensions, when there is an additional internal state variable for these particles, it will make a difference whether or not that internal state variable corresponds to a continuous degree of freedom, that difference relating to the possibility of parastatistics. The third question was alluded to in the opening of this article. Given an internal SO(3) variable for relativistic particles is there any argument that would now connect spin and statistics?

The author thanks J Avron and B Gaveau for helpful discussions. This work was supported in part by NSF grant PHY 90-15858.

## References

[1] Schulman L S 1981 *Techniques and Applications of Path Integration* (New York: Wiley)
[2] Laidlaw M G G and DeWitt-Morette C 1971 *Phys. Rev. D* **3** 1375
[3] Bourdeau M and Sorkin R D 1992 *Phys. Rev.* D **45** 687
[4] Schulman L S 1968 *Phys. Rev.* **176** 1558
[5] —— 1971 *J. Math. Phys.* **12** 304
[6] Canright G S and Girvin S M 1990 *Science* **247** 1197
[7] Schulman L S 1975 *Functional Integration and Its Applications* ed A M Arthurs (Oxford: Oxford University Press) p 144
[8] Schulman L S 1970 *Nucl. Phys.* B **18** 595. See also Schulman L S 1967 *A Path Integral for Spin*, PhD thesis Princeton University
[9] Bopp F and Haag R 1950 *Z. Naturforsch.* A **5** 644
[10] Rosen N 1951 *Phys. Rev.* **82** 621